GENERAL RELATIVITY ON A NUL SURFACE: HAMILTONIAN FORMULATION IN THE TELEPARALLEL GEOMETRY

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Abstract

The Hamiltonian formulation of general relativity on a null surface is established in the teleparallel geometry. No particular conditions on the tetrads are imposed, such as the time gauge condition. By means of a 3+1 decomposition the resulting Hamiltonian arises as a completely constrained system. However, it is structurally different from the standard Arnowitt-Deser-Misner (ADM) type formulation. In this geometrical framework the basic field quantities are tetrads that transform under the global SO(3,1) and the torsion tensor.

PACS numbers: 04.20.Cv, 04.20.Fy, 04.90.+e
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I. Introduction

The study of asymptotically flat gravitational waves is an important and interesting issue in general relativity. It started with the pioneering work of Bondi[1], which was subsequently generalized by Sachs[2]. It was soon realized that the description of gravitational waves on a null surface facilitates the characterization of the true, independent degrees of freedom of the gravitational field. This characterization may possibly be mandatory to the quantization of gravity. Moreover, null surfaces play an important role in the study of gravitational radiation.

Difficulties in working with the dynamics of null surfaces are well known. The latter are characterized by the condition $g^{00} = 0$. However, if we naively impose this condition in Einstein's equations we spoil the six evolution equations, since these equations become exempt of second order time derivatives and consequently the evolution becomes undetermined. Imposition of the above condition in the variation of the Hilbert-Einstein action integral leads to nine equations only. Therefore attempts have been made to arrive at a well posed *characteristic* initial value problem.

The analysis of the initial value problem for asymptotically flat, nonradiating space-times is reasonably well understood. Moreover, the Arnowitt-Deser-Misner (ADM) Hamiltonian formulation[3] is usually taken as a paradigm to the study of the dynamics of spacelike surfaces. In contrast, there does not seem to exist a widely accepted formulation of the characteristic initial value problem, or of the corresponding Hamiltonian formulation, as we observe from the vast literature on the subject. The initial value problem has been analysed, for instance, in [4, 5, 6, 7, 8], whereas the Hamiltonian formulation has been investigated both in a 2+2 decomposition ([9, 10]) and in a 3+1 decomposition ([11, 12, 13, 14]). In particular, the work of refs.[10, 13, 14] is developed in the context of Ashtekar variables. While all of these approaches add some progress to the understanding of the dynamics of the gravitational field on null surafaces, we see that at the present time there does not exist a definite, irrefutable Hamiltonian formulation which would, accodding to Goldberg *et. al.*[13], display in an isolated form the true degrees of freedom and the observables of the theory, in such a way that the dynamics of these degrees of freedom is singled out from the dynamics of the remaining field quantities.

In this paper we construct the Hamiltonian for the gravitational field on

a null surface in the teleparallel geometry. The analysis of the dynamics of spacelike surfaces in this geometry has already been carried out in [15]. However in that analysis the time gauge condition was imposed in order to simplify the considerations. Since we cannot impose at the same time the null surface and the time gauge conditions, the problem has to reconsidered in a new fashion.

The analysis of the gravitational field in this geometrical framework has proven to be useful, among other reasons because of the appearance of a scalar density in the form of a divergence in the Hamiltonian constraint, and which is identified as the gravitational energy density[16]. This expression for the gravitational energy can be applied to concrete, physical configurations of the gravitational field (see, for instance, refs.[16, 17, 18, 19]). In this paper we obtain a similar structure. The four constraints of the theory contain each one a divergence which altogether constitute a vector density, and which strongly suggests a definition for the gravitational radiation energy. The detailed analysis of this issue is not carried out here.

One achievement of this long term program is to demonstrate that general relativity can be alternatively presented and discussed in the teleparallel geometry, without recourse to the Riemann curvature tensor or to the Levi-Civita (metric compatible) connection. In this sense, this geometrical framework allows an alternative understanding of the gravitational field.

In section II we present the Lagrangian formulation of the teleparallel equivalent of general relativity (TEGR) in a way somewhat different from what has been presented so far. In [15] the theory is formulated initially with a local SO(3,1) symmetry, and in the Hamiltonian analysis, after fixing the time gauge condition, it is concluded that in order to arrive at a set of first class constraints it is necessary to transform the SO(3,1) into a global symmetry group. In this paper the symmetry group is taken as the global SO(3,1) from the outset. In section III we present the boundary conditions for the tetrad components, assuming that the radiation is due to a localized source. In section IV we present in detail the construction of the Hamiltonian, obtained by a 3+1 decomposition. In the last section we present additional comments and point out further developments.

Notation: spacetime indices $\mu, \nu, ...$ and SO(3,1) indices $a, b, ...$ run from 0 to 3. In the 3+1 decomposition latin indices from the middle of the alphabet indicate space indices according to $\mu = 0, i, \quad a = (0), (i)$. The tetrad field

$e^a{}_\mu$ yields the usual definition of the torsion tensor: $T^a{}_{\mu\nu} = \partial_\mu e^a{}_\nu - \partial_\nu e^a{}_\mu$. The flat, Minkowski spacetime metric is fixed by $\eta_{ab} = e_{a\mu} e_{b\mu} g^{\mu\nu} = (-+++)$.

II. The Lagrangian formulation of the TEGR

In [15] the Lagrangian formulation of the TEGR is presented in terms of the tetrad field and a spin connection $\omega_{\mu ab}$. Both quantities transform under the local SO(3,1) group but are not related, not even by the field equations. The equivalence of the teleparallel Lagrangian with the Hilbert-Einstein Lagrangian holds provided we require the vanishing of the curvature tensor tensor $R^a{}_{b\mu\nu}(\omega)$. In the Hamiltonian analysis we conclude that the symmetry group must be the global SO(3), and eventually the connection is discarded.

In this paper we will establish the Lagrangian density in terms of the tetrad field only. The symmetry group is the global SO(3,1). The Lagrangian density is given by

$$L(e) \;=\; -k\, e\, \Sigma^{abc} T_{abc} \;, \tag{1}$$

where $k = \frac{1}{16\pi G}$, G is Newton's constant, $e = det(e^a{}_\mu)$, $T_{abc} = e_b{}^\mu e_c{}^\nu T_{a\mu\nu}$ and

$$\Sigma^{abc} \;=\; \frac{1}{4}(T^{abc} + T^{bac} - T^{cab}) + \frac{1}{2}(\eta^{ac} T^b - \eta^{ab} T^c) \;. \tag{2}$$

Tetrads transform space-time into SO(3,1) indices and vice-versa. The trace of the torsion tensor is given by

$$T_b = T^a{}_{ab} \;.$$

The tensor Σ^{abc} is defined such that

$$\Sigma^{abc} T_{abc} \;=\; \frac{1}{4} T^{abc} T_{abc} + \frac{1}{2} T^{abc} T_{bac} - T^a T_a \;.$$

The field equations obtained from (1) read

$$\frac{\delta L}{\delta e^{a\mu}} \;=\; e_{a\lambda} e_{b\mu} \partial_\nu (e \Sigma^{b\lambda\nu}) - e\left(\Sigma^{b\nu}{}_a T_{b\nu\mu} - \frac{1}{4} e_{a\mu} T_{bcd} \Sigma^{bcd}\right) \;=\; 0 \;. \tag{3}$$

It can be shown by explicit calculations[15] that these equations yield Einstein's equations:

$$\frac{\delta L}{\delta e^{a\mu}} \equiv \frac{1}{2} e \left\{ R_{a\mu}(e) - \frac{1}{2} e_{a\mu} R(e) \right\} .$$

In order to obtain the canonical formulation we need a first order differential formulation of (1). This is easily obtanained through the introduction of an auxiliary field quantity $\phi_{abc} = -\phi_{acb}$. The first order differential formulation of the TEGR is described by the following Lagrangian density,

$$L(e, \phi) = k \, e \, \Lambda^{abc} (\phi_{abc} - 2T_{abc}) , \tag{4}$$

where Λ^{abc} is defined in terms of ϕ^{abc} exactly as Σ^{abc} in terms of T^{abc}:

$$\Lambda^{abc} = \frac{1}{4}(\phi^{abc} + \phi^{bac} - \phi^{cab}) + \frac{1}{2}(\eta^{ac}\phi^{b} - \eta^{ab}\phi^{c}) . \tag{5}$$

Variation of the action constructed out of (4) with respect to ϕ^{abc} yields

$$\Lambda_{abc} = \Sigma_{abc} , \tag{6}$$

which, after some manipulations, can be reduced to

$$\phi_{abc} = T_{abc} . \tag{7}$$

The equation above may be split into two equations:

$$\phi_{a0k} = T_{a0k} = \partial_0 e_{ak} - \partial_k e_{a0} , \tag{8a}$$

$$\phi_{aik} = T_{aik} = \partial_i e_{ak} - \partial_k e_{ai} . \tag{8b}$$

Taking into account eq. (7), it can be shown that the second field equation, the variation of the action integral with respect to $e_{a\mu}$, leads precisely to (3). Therefore (1) and (4) exhibit the same physical content.

In section IV we will make explicit reference to null surfaces. The theory defined by (1) or (4) describes an arbitrary gravitational field, as there is no restriction in the form of a Lagrange multiplier fixing some particular geometry. Without going into details we just mention that if we impose the condition $g^{00} = 0$ in (3) the resulting equation will still have second order

time derivatives (note that this equation has one SO(3,1) and one space-time index).

Before closing this section let us make a remark. The theory defined by (1) and (2) has been considered in the literature, in a different context, as the translational gauge formulation of Einstein's general relativity[20]. It is argued in this approach that (1) is invariant under *local* SO(3,1) transformations up to a total divergence. This divergence is then discarded, from what is concluded that (1) exhibits local gauge symmetry. We do not endorse this point of view. A careful analysis of this divergence (the last term of eq. (12) of [20]) shows that in general it does not vanish for arbitrary elements of the SO(3,1) group when integrated over the whole spacelike surface. Problems arise if the SO(3,1) group elements fall off as *const.* + $O(\frac{1}{r})$ when $r \to \infty$. Therefore the action is not, in general, invariant under such transformations. Surface terms play a very important role in action integrals for the gravitational field, so that one cannot *arbitrarily* add or remove them. Moreover, if (1) were actually invariant under the local SO(3,1) group, then the theory would have six additional constraints, which would spoil the counting of degrees of freedom of the theory (see eqs. (18), (19) and (31) ahead).

III. The boundary conditions

In order to guarantee that the space-time of a localized radiating source is asymptotically flat we adopt the conditions laid down by Bondi[1] and Sachs[2] for the metric tensor. The conditions on the tetrads are simply obtained by constructing the tetrads associated with these radiating fields and identifying the asymptotic behaviour when $r \to \infty$. Of course there is an infinity of tetrads that yield the same metric tensor. However, we will consider a typical configuration and assume the generality of our considerations. For simplicity we will consider in detail Bondi's metric.

Bondi's metric is not an exact solution of Einstein's equations. In terms of radiation coordinates (u, r, θ, ϕ), where u is the retarded time and r is a luminosity distance, Bondi's radiating metric is written as

$$ds^2 \;=\; -\Big(\frac{V}{r}e^{2\beta} - U^2\,r^2 e^{2\gamma}\Big)du^2 - 2e^{2\beta}du\,dr - 2U\,r^2\,e^{2\gamma}du\,d\theta$$

$$+r^2\Big(e^{2\gamma}\,d\theta^2+e^{-2\gamma}\,sin^2\theta\,d\phi^2\Big)\,. \tag{9}$$

The metric above is such that the surfaces for which $u = constant$ are null hypersurfaces. Each null radial (light) ray is labelled by particular values of u,θ and ϕ. At spacelike infinity u takes the standard form $u = t - r$. The four quantities appearing in (9), V, U, β and γ are functions of u, r and θ. Thus (9) displays axial symmetry. A more general form of this metric has been given by Sachs[2], who showed that the most general metric tensor describing asymptotically flat gravitational waves depends on six functions of the coordinates.

The functions in (9) satisfy the following asymptotic behaviour:

$$\beta \;=\; -\frac{c^2}{4r^2}+...$$

$$\gamma \;=\; \frac{c}{r}+...$$

$$\frac{V}{r} \;=\; 1-\frac{2M}{r}-\frac{1}{r^2}\Big[\frac{\partial d}{\partial\theta}+d\,cot\theta-\Big(\frac{\partial c}{\partial\theta}\Big)^2-4c\Big(\frac{\partial c}{\partial\theta}\Big)cot\theta-\frac{1}{2}\,c^2\Big(1+8cot^2\theta\Big)\Big]+...$$

$$U \;=\; \frac{1}{r^2}\Big(\frac{\partial c}{\partial\theta}+2c\,cot\theta\Big)+\frac{1}{r^3}\Big(2d+3c\,\frac{\partial c}{\partial\theta}\,cot\theta+4c^2\,cot\theta\Big)+...\,,$$

where $M = M(u,\theta)$ and $d = d(u,\theta)$ are the mass aspect and the dipole aspect, respectively, and from the function $c(u,\theta)$ we define the news function $\frac{\partial c(u,\theta)}{\partial u}$.

One possible realization of this metric tensor in terms of tetrad fields is given by

$$e_{a\mu}=\begin{pmatrix} -e^{\beta}(\frac{V}{r})^{\frac{1}{2}} & -e^{\beta}(\frac{V}{r})^{-\frac{1}{2}} & 0 & 0 \\ -r\,U\,e^{\gamma}\,cos\theta\,cos\phi & e^{\beta}(\frac{V}{r})^{-\frac{1}{2}}sin\theta\,cos\phi & r\,e^{\gamma}\,cos\theta\,cos\phi & -r\,e^{-\gamma}\,sin\theta\,sin\phi \\ -r\,U\,e^{\gamma}\,cos\theta\,sin\phi & e^{\beta}\,(\frac{V}{r})^{-\frac{1}{2}}sin\theta\,sin\phi & r\,e^{\gamma}\,cos\theta\,sin\phi & r\,e^{-\gamma}\,sin\theta\,cos\phi \\ r\,U\,e^{\gamma}\,sin\theta & e^{\beta}\,(\frac{V}{r})^{-\frac{1}{2}}cos\theta & -r\,e^{\gamma}\,sin\theta & 0 \end{pmatrix}. \tag{10}$$

From the expression above we obtain the asymptotic behaviour of the tetrad components in cartesian coordinates:

$$e_{(0)0} \sim 1 + O(\frac{1}{r}) + ... \,, \tag{11a}$$

$$e_{(0)k} \sim 1 + O(\frac{1}{r}) + ... \,, \tag{11b}$$

$$e_{(i)0} \sim O(\frac{1}{r}) + ... \,, \tag{11c}$$

$$e_{(i)k} \sim \delta_{ik} + \frac{1}{2}h_{ik}(\frac{1}{r}) + ... \,. \tag{11d}$$

These expressions establish the boundary conditions for the tetrads. As a final comment, we remark that if we make $M = d = 0$ in (9), Bondi's metric reduces to the flat space-time metric in radiation coordinates, and so does expression (10) for the tetrads. It can be shown that in this case all components of the torsion tensor vanish.

IV. The 3+1 decomposition

There are several fundamental differences between the analysis of this section and the approach of Goldberg *et. al.*[13, 14]. In the latter, complex valued field variables and an orthonormal set of null vectors adapted to a null surface are employed. In contrast, we adopt ordinary, real valued tetrads. Nevertheless, the present analysis is conceptually the same as that developed in [13, 14]. We conclude, however, that it is unecessary to establish a 3+1 decomposition for the tetrads, as it is normally done. The Hamiltonian formulation arises naturally in terms of the four dimensional tetrad field and its inverse, as we will see.

The Hamiltonian formulation is established from the first order differential Lagrangian density (4). Space and time derivatives appear only in T_{abc}. Expression (4) can be rewritten as

$$L(e,\phi) = -4ke\,\Lambda^{a0k}\,\dot{e}_{ak} + 4ke\,\Lambda^{a0k}\,\partial_k e_{a0} - 2ke\,\Lambda^{aij}\,T_{aij} + ke\Lambda^{abc}\,\phi_{abc} \,, \tag{12}$$

where the dot indicates time derivative, and

$$\Lambda^{a0k} \;=\; \Lambda^{abc}\, e_b{}^0\, e_c{}^k \;,$$

$$\Lambda^{aij} \;=\; \Lambda^{abc}\, e_b{}^i\, e_c{}^j \;.$$

Thus the momentum canonically conjugated to e_{ak} is given by

$$\Pi^{ak} \;=\; -4k\, e\, \Lambda^{a0k} \;, \tag{13}$$

Expression (12) is then rewritten as

$$L \;=\; \Pi^{ak}\, \dot{e}_{ak} - \Pi^{ak}\, \partial_k e_{a0} - 2ke\, \Lambda^{aij}\, T_{aij} + ke\, \Lambda^{abc}\, \phi_{abc} \;. \tag{14}$$

In order to establish the Hamiltonian formulation we need to rewrite the expression above in terms of e_{ak}, Π^{ak} and further nondynamical quantities. However this is not a trivial procedure. In [15] the 3+1 decomposition of the theory was possible, to a large extent because of the time gauge condition $e_{(i)}{}^0 = e^{(0)}{}_k = 0$. This condition resulted in a tremendous simplification of the analysis. It is clear that we cannot impose simultaneously the time gauge condition and the null surface condition. Therefore the present analysis will be totally different from that of [15].

The construction can be formally carried out in two steps. First, we substitute the Lagrangian field equation (8b) into (14), so that half of the auxiliary fields, ϕ_{aij}, are eliminated from the Lagrangian. Second, we should be able to express the remaining auxiliary fields, ϕ_{a0k}, in terms of the momenta Π^{ak}. This is a nontrivial step.

We need to work out the explicit form of Π^{ak}. It is given by

$$\Pi^{ak} \;=\; k\, e \Big\{ g^{00}(-g^{kj}\phi^a{}_{0j} - e^{aj}\phi^k{}_{0j} + 2e^{ak}\phi^j{}_{0j})$$

$$+g^{0k}(g^{0j}\phi^a{}_{0j} + e^{aj}\phi^0{}_{0j}) + e^{a0}(g^{0j}\phi^k{}_{0j} + g^{kj}\phi^0{}_{0j}) - 2(e^{a0}g^{0k}\phi^j{}_{0j} + e^{ak}g^{0j}\phi^0{}_{0j})$$

$$-g^{0i}g^{kj}\phi^a{}_{ij} + e^{ai}(g^{0j}\phi^k{}_{ij} - g^{kj}\phi^0{}_{ij}) + 2(g^{0i}e^{ak} - g^{ik}e^{a0})\phi^j{}_{ij} \Big\} \;. \tag{15}$$

From now on we impose the null surface condition

$$g^{00} \;=\; 0\;.$$

The imposition of this condition at the end of the Legendre transform would render infinities. Denoting (..) and [..] as the symmetric and anti-symmetric parts of field quantities, respectively, we can decompose Π^{ak} into irreducible components:

$$\Pi^{ak} \;=\; e^a{}_i\,\Pi^{(ik)} + e^a{}_i\,\Pi^{[ik]} + e^a{}_0\,\Pi^{0k}\;, \tag{16}$$

where

$$\Pi^{(ik)} \;=\; k\,e\Big\{g^{0k}(g^{0j}\phi^i{}_{0j} + g^{ij}\phi^0{}_{0j} - g^{0i}\phi^j{}_{0j})$$

$$+g^{0i}(g^{0j}\phi^k{}_{0j} + g^{kj}\phi^0{}_{0j} - g^{0k}\phi^j{}_{0j}) - 2g^{ik}\,g^{0j}\phi^0{}_{0j} \;+\; \Delta^{ik}\Big\}\;, \tag{17a}$$

$$\Delta^{ik} \;=\; -g^{0m}(g^{kj}\phi^i{}_{mj} + g^{ij}\phi^k{}_{mj} - 2g^{ik}\phi^j{}_{mj}) - (g^{km}g^{0i} + g^{im}g^{0k})\phi^j{}_{mj}\;, \tag{17b}$$

$$\Pi^{[ik]} \;=\; k\,e\Big\{-g^{im}g^{kj}\phi^0{}_{mj} + (g^{im}g^{0k} - g^{km}g^{0i})\phi^j{}_{mj}\Big\} \;\equiv\; k\,e\,p^{ik}\;, \tag{18}$$

$$\Pi^{0k} \;=\; -2k\,e\,(g^{kj}g^{0i}\phi^0{}_{ij} - g^{0k}g^{0i}\phi^j{}_{ij}) \;\equiv\; k\,e\,p^k\;. \tag{19}$$

The crucial observation of this analysis is that only $\Pi^{(ik)}$ depends on the "velocities" $\phi^a{}_{0j}$. $\Pi^{[ik]}$ and Π^{0k} depend solely on $\phi^a{}_{ij} = T^a{}_{ij}$. Therefore we can express only six of the "velocity" fields $\phi^a{}_{0j}$ in terms of the momenta $\Pi^{(ik)}$. In order to find out which components of $\phi^a{}_{0j}$ can be inverted we decompose the latter identically as

$$\phi^a{}_{0j} \;=\; e^{ai}\,\psi_{ij} + e^{ai}\,\sigma_{ij} + e^{a0}\,\lambda_j\;, \tag{20}$$

with the following definitions:

$$\psi_{ij} \;=\; \psi_{ji} \;=\; \frac{1}{2}(\phi_{i0j} + \phi_{j0i})\;,$$

$$\sigma_{ij} \;=\; -\sigma_{ji} \;=\; \frac{1}{2}(\phi_{i0j} - \phi_{j0i}) \;,$$

$$\lambda_j \;=\; \phi_{00j}$$

Substituting (20) in (17a) we find that $\Pi^{(ik)}$ depends only on ψ_{ij}:

$$\Pi^{(ik)} \;=\; k\, e\Big\{2(\, g^{0k}g^{im}g^{0j}\psi_{mj} + g^{0i}g^{km}g^{0j}\psi_{mj} - g^{0i}g^{0k}g^{mn}\psi_{mn} - g^{ik}g^{0m}g^{0n}\psi_{mn})$$

$$+\Delta^{ik}\Big\} \;. \tag{21}$$

Therefore if terms like σ_{ij} and λ_j appear in L, other than in $\Pi^{ak}\dot{e}_{ak}$, we would have difficulties in performing the Legendre transform, because they cannot be transformed into any momenta ($\Pi^{[ik]}$ and Π^{0k} do not depend on them). Fortunately, they do not appear. Let us rewrite L given by (14) in terms of (15) and (20), assuming from now on that $\phi^a{}_{ij} = T^a{}_{ij}$. It is given by

$$L \;=\; \Pi^{ak}\dot{e}_{ak} + e_{a0}\,\partial_k\Pi^{ak} - \partial_k(e_{a0}\Pi^{ak})$$

$$+k\, e(-\frac{1}{4}g^{im}g^{nj}T^a{}_{mn}T_{aij} - \frac{1}{2}g^{jn}T^i{}_{mn}T^m{}_{ij} + g^{ik}T^j{}_{ji}T^n{}_{nk})$$

$$-\frac{1}{2}\phi_{a0k}\Big\{\Pi^{ak} + ke[g^{0i}g^{jk}T^a{}_{ij} - e^{ai}(g^{0j}T^k{}_{ij} - g^{jk}T^0{}_{ij}) - 2(e^{ak}g^{0i} - e^{a0}g^{ki})T^j{}_{ij}]\Big\} \;. \tag{22}$$

The field ϕ_{a0k} appears only in the last line of the expression above. The terms that appear together with Π^{ak} in this line exactly subtract the last line of (15). It is possible to check by explicit calculations that the last line of (22) can be written as

$$-\frac{1}{2}\psi_{ik}\,(\Pi^{(ik)} - k\, e\,\Delta^{ik}) \;. \tag{23}$$

We can then proceed and complete the Legendre transform. In the present case the latter amounts to expressing ψ_{ik} in terms of $\Pi^{(ij)}$. The inversion can be made and leads to

$$\psi_{ik} \;=\; \frac{1}{2}\Big\{g_{0m}(g_{0i}g_{jk}+g_{0k}g_{ji})P^{mj} - \frac{1}{2}(g_{0i}g_{0k}g_{mn}+g_{0m}g_{0n}g_{ik})P^{mn}\Big\}\;, \quad (24)$$

where

$$P^{ik} \;=\; \frac{1}{ke}\Pi^{(ik)} - \Delta^{ik}\;. \quad (25)$$

Substituting (23) and (24) back in (22) we finally arrive at the primary Hamiltonian $H_0 = p\dot{q} - L$:

$$H_0 \;=\; -e_{a0}\,\partial_k\Pi^{ak} + k\,e\,(\,\frac{1}{4}g^{im}g^{nj}T^a{}_{mn}T_{aij} + \frac{1}{2}g^{jn}T^i{}_{mn}T^m{}_{ij} - g^{ik}T^j{}_{ji}T^n{}_{nk}\,)$$

$$+\,ke\frac{1}{2}(g_{0i}g_{0m}g_{nk} - \frac{1}{2}g_{0i}g_{0k}g_{mn})P^{mn}P^{ik}\;. \quad (26)$$

Since equations (18) and (19) constitute primary constraints, they have to be added to H_0, and so the Hamiltonian becomes

$$H \;=\; H_0 \;+\; \alpha_{ik}\,(\Pi^{[ik]} - k\,e\,p^{ik}) \;+\; \beta_k\,(\Pi^{0k} - k\,e\,p^k) \;+\; \gamma\,g^{00} \;+\; \partial_k(e_{a0}\Pi^{ak})\;. \quad (27)$$

The quantities α_{ik}, β_k and γ are Lagrange multipliers.

Next we note that since the momenta $\{\Pi^{a0}\}$ are identically vanishing, they also constitute primary constraints, which induce the secondary constraints

$$C^a \;\equiv\; \frac{\delta H}{\delta e_{a0}} \;=\; 0\;. \quad (28)$$

In the proccess of varying H with respect to e_{a0} we only have to consider H_0, because variation of the constraints lead to the contraints themselves:

$$\frac{\delta}{\delta e_{a0}}\,(\Pi^{[ik]} - k\,e\,p^{ik}) \;=\; -\frac{1}{2}\Big(e^{ai}(\Pi^{0k} - k\,e\,p^k) - e^{ak}(\Pi^{0i} - k\,e\,p^i)\Big)\;, \quad (29)$$

$$\frac{\delta}{\delta e_{a0}}\,(\Pi^{0k} - k\,e\,p^k) \;=\; -e^{a0}\,(\,\Pi^{0k} - k\,e\,p^k\,)\;. \quad (30)$$

As we will see, the evaluation of the constraints C^a according to (28) reveals the constraint structure of the Hamiltonian H_0. After a long calculation we arrive at

$$
\begin{aligned}
C^a \;&=\; -\partial_k \Pi^{ak} + ke\, e^{a0}\Big\{\frac{1}{4} g^{im} g^{nj} T^b{}_{mn} T_{bij} + \frac{1}{2} g^{jn} T^i{}_{mn} T^m{}_{ij} - g^{ik} T^j{}_{ji} T^n{}_{nk} \\
&\quad + \frac{1}{2} g_{0i} (g_{0m} g_{nk} - \frac{1}{2} g_{0k} g_{mn}) P^{mn} P^{ik}\Big\} - ke\, e^{ai}\Big\{ g^{0m} g^{nj} T^b{}_{ij} T_{bmn} \\
&\quad + g^{0j} T^m{}_{ni} T^n{}_{mj} + g^{nj} T^0{}_{mn} T^m{}_{ij} - 2 g^{0k} T^j{}_{ji} T^n{}_{nk} - 2 g^{jk} T^0{}_{ij} T^n{}_{nk}\Big\} \\
&\quad + ke\Big\{ e^{aj} g_{ij} (g_{0m} g_{nk} - \frac{1}{2} g_{0k} g_{mn}) P^{mn} P^{ik} \\
&\quad + \frac{1}{2} g_{0i} (g_{0m} g_{nk} - \frac{1}{2} g_{0k} g_{mn}) (P^{mn} \gamma^{aik} + P^{ik} \gamma^{amn})\Big\}\;. \qquad (31)
\end{aligned}
$$

The quantity γ^{aik} appearing in (31) is defined by

$$
\gamma^{aik} \;=\; e^{al}\Big\{ g^{0j} (g^{0k} T^i{}_{lj} + g^{0i} T^k{}_{lj}) - 2 g^{0i} g^{0k} T^j{}_{lj} + (g^{kj} g^{0i} + g^{ij} g^{0k} - 2 g^{ik} g^{0j}) T^0{}_{lj}\Big\}\;. \qquad (32)
$$

We immediately note that C^a satisfies the relation

$$
e_{a0} C^a \;=\; H_0\;. \qquad (33)
$$

Therefore we can write the final form of the completely constrained Hamiltonian:

$$
H \;=\; e_{a0} C^a \;+\; \alpha_{ik}\,(\Pi^{[ik]} - kep^{ik}) \;+\; \beta_k\,(\Pi^{0k} - kep^k) \;+\; \gamma\, g^{00} \;+\; \partial_k (e_{a0} \Pi^{ak})\;. \qquad (34)
$$

Although e_{a0} appears as a Lagrange multiplier, it is also contained both in H_0 and in C^a. However, it is possible to check that e_{a0} is a true Lagrange

multiplier. By just making use of the orthogonality relations of the tetrads, it is possible to verify that the constraints C^a satisfy the relation

$$e_{a0}\,\frac{\delta C^a}{\delta e_{b0}} \;=\; 0\;,$$

from what we conclude that variation of H given by (34) with respect to e_{a0} yields C^a plus the constraints on the right hand side of (29) and (30).

V. Comments

In the last section we have completed the 3+1 decomposition of the Lagrangian density (4) on a null surface. In this procedure all tetrad and metric components are four dimensional quantities. We have not established any decomposition for these fields, basically because it was not needed. Since the tetrads do not obey any particular gauge condition, the nondynamical component $e_{(0)0}$ cannot be identified with the usual lapse function.

The final form of the Hamiltonian, eq.(34), is written as a sum of the constraints of the theory. One major difference between this Hamiltonian formulation and the ADM-type formulation is that in the latter the usual vector constraint H_i is linear in the momenta[3], whereas here both $C^{(i)}$ and $C^{(0)}$ are linear and quadratic in $\Pi^{(ik)}$, in general.

The next step is the determination of the constraint algebra. The algebra of the ten constraints, equations (18), (19) and (31), is expected to be quite intricate. The analysis of [13, 14] showed the existence of second class constraints. It is likely the same complication arises here. This issue will be investigated in the near future.

As we mentioned in the introduction, one major motivation for the present analysis is the establishment of an expression for the gravitational energy-momentum vector density. In the present case this expression is restricted to configurations of the gravitational field that describe gravitational waves. Our previous experience on this subject lead us to conclude that the covariant gravitational energy-momentum P^a is given by

$$P^a \;=\; -\int_V d^3x\,\partial_k \Pi^{ak}\;. \tag{35}$$

As before[16], the integral form of the constraint $C^{(0)} = 0$ can be interpreted as an energy equation of the type $H - E = 0$. Expression (35) allows us to compute the energy-momentum of the gravitational radiation field for an arbitrary volume of the three-dimensional space. This analysis will be carried out in the context of the Bondi and Sachs metrics and presented in detail elsewhere.

Acknowledgements
This work was supported in part by CNPQ. J. F. R. N. is supported by CAPES, Brazil.

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